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Structure of Right Subdirectly Irreducible Rings I*

MADHUKAR G. DESHPANDE

Department of Mathematics, Marquette University, Milwaukee, Wisconsin 53233

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INTRODUCTION

A module M_R is said to be subdirectly irreducible if the intersection M_0 of all nonzero submodules of M is nonzero, and in this case M_0 is called the heart of M. This definition of subdirect irreducibility is equivalent to that of Birkhoff who introduced the general notion of subdirect product of universal algebras in [1]. We define a ring R to be right subdirectly irreducible (RSI for brevity), if R_R is a subdirectly irreducible module.

Suppose R is a RSI ring with heart H. Then H is the unique minimal right ideal of R. For any $a \in R$, the mapping $h \to ah$ of H onto aH implies that either $aH = 0$ or aH is a minimal right ideal and, hence, equal to H. Thus $RH \subseteq H$ and, in the important case when R has an identity, $RH = H$. In any case, the heart is a two-sided ideal. Consequently, a RSI ring R is always subdirectly irreducible in the sense that the intersection of all nonzero two-sided ideals is nonzero. The converse of this is not true as can be seen from the example of any $n \times n$ ($n \geqslant 2$) matrix ring over a division ring. Obviously enough, for a commutative ring the two notions coincide.

Commutative subdirectly irreducible rings have been studied by McCoy [9] and by Divinsky [4]. In this paper, a characterization of RSI rings with d.c.c. on right ideals (Theorem 3.1) is given similar to that of McCoy [9, page 382] for the commutative case, and it is proved, along with some other equivalent conditions (Theorem 2.1), that a right noetherian RSI ring is right Artinian iff the left annihilator of the heart is a maximal right ideal. Example 2.2 shows that right noetherian RSI rings need not be right Artinian. Conditions are then obtained in Section 4 under which a subdirectly irreducible module is completely indecomposable in the sense of [5, p. 349]. As a consequence of this we deduce that RSI rings with a.c.c. on left and right

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ideals satisfy d.c.c. on left and right ideals. Example 4.5 shows that a RSI ring with d.c.c. on left and right ideals need not be completely indecomposable. It also shows that a noetherian ring in which every right ideal is an annihilator need not be self-injective; thus answering a question raised by Cohn in [3, p. 75].

1. Notation and Preliminaries

All rings considered here have an identity and all modules are unitary. Unless otherwise stated, by a molule M, we mean a right R-module over a ring R. If R is a RSI ring, H will always denote the heart of R. If h is a nonzero element of H, $h^l = \{a \in R : ah = 0\}$ will be denoted by N. Since $H = hR$, N is also equal to H^l and hence is a two-sided ideal of R. Let $D = \mathrm{Hom}_R(H_R, H_R)$. Then by Schur's lemma, D is a division ring. If $\hat{R}$ denotes the injective hull of R_R and K the ring of endomorphism $\mathrm{Hom}_R(\hat{R}, \hat{R})$ of $\hat{R}$, then we can regard ${}_K\hat{R}_R$ a bimodule as in [8, p. 94]. Let L denote the ideal of K defined by $\{f \in K : f^r \text{ large in } \hat{R}\}$; where, for any $f \in K$, f^r denotes the kernel of the homomorphism f. We will use the symbols $H, N, D, \hat{R}, K, L$ always in the sense described above.

If R is RSI, then H_R is a large submodule of R_R. Since R_R is large in $\hat{R}$, it follows from [8, Example 3, p. 62] that H_R is large in $\hat{R}$. H is also a minimal submodule of $\hat{R}$ and thus $\hat{R}_R$ is subdirectly irreducible with heart H. A subdirectly irreducible module is in particular a uniform module and in a uniform module a submodule is large iff it is nonzero. It can now be seen that N and L defined above are also characterised by

1.2. $\qquad N = \{a \in R : a^r \neq 0\} \qquad$ and $\qquad L = \{f \in K : f^r \neq 0\}$,

where for any x acting from the left on R or $\hat{R}$, x^r is the right annihilator of x in R or $\hat{R}$, respectively.

In a RSI ring R, H^2 must either equal 0 or H because $H^2 \subseteq H$. The following proposition shows that in the case when $H^2 = H$, R is a division ring.

1.3. Proposition. *If R is* RSI *and $H^2 = H$, then R is a division ring.*

For, since H is a minimal right ideal and $H^2 \neq 0$, by Proposition 1 of [8, p. 62], $H = eR$ for some idempotent e in R. Since $1 - e$ is also an idempotent such that $eR \cap (1 - e)R = 0$, we must have $(1 - e)R = 0$, i.e., $e = 1$. Thus $H = R$, and R has no proper nontrivial right ideal which implies that R is a division ring.

Since a RSI ring R is in particular a right uniform ring, we can apply

Theorem 3.1 of [6, p. 131] and conclude that R/N is an integral domain and in addition, if R is right noetherian, then N coincides with the set of nilpotent elements of R. Also, by using Examples 2 and 3 of [8, p. 104], we note that K is a local ring with L as its unique maximal right ideal. Thus K/L is a division ring.

1.4. THEOREM. *If R is* RSI, *then R/N can be embedded isomorphically into the division ring D and $D \simeq K/L$.*

Proof. For any a in R, $f_a : H \to H$ defined by $f_a(h) = ah$ for all h in H is an element of D. The mapping $f : R \to D$ given by $f(a) = f_a$ is a ring homomorphism with ker N and, accordingly, f induces an isomorphism of R/N onto a subring of D.

Now, let $d : H \to H$ be an arbitrary element of D. By injectivity of $\hat{R}$, d can be extended to an element $d^* : \hat{R} \to \hat{R}$ of K. Also, if d^* and d^{**} are two extensions of d, then $H \subseteq \ker(d^* - d^{**})$ and thus $d^* - d^{**} \in L$. If $\theta : D \to K/L$ is defined by $\theta(d) = d^* + L$, then it can be verified that θ is a homomorphism of D onto K/L. It is also one-one because if $f \in L$, then $\ker f \neq 0$, and since H is contained in every nonzero submodule of $\hat{R}$, f restricted to H must be a zero map.

2. RINGS WITH A.C.C. ON RIGHT IDEALS

We have already remarked that if R is a right noetherian RSI ring, then N coincides with the set of nilpotent elements and hence it is the nil radical of R. By a.c.c., N is then itself nilpotent.

2.1 THEOREM. *Let R be a* RSI *ring with* a.c.c. *on right ideals. Then the following are equivalent and any of them implies that R is a local ring.*

(i) *R has d.c.c. on right ideals*;

(ii) *N is a maximal right ideal*;

(iii) *Rh is a minimal left ideal for every $h \in H$, $h \neq 0$*;

(iv) *$a^r = 0$ implies $a^l = 0$ for any $a \in R$, i.e., every right regular element of R is left regular.*

(v) *$h^r = N$ for every $h \in H$, $h \neq 0$.*

(vi) *$R/N \simeq D$.*

Proof. The proof will consist of showing the implications

$$(\mathrm{i}) \Rightarrow (\mathrm{iv}) \Rightarrow (\mathrm{v}) \Rightarrow (\mathrm{vi}) \Rightarrow (\mathrm{iii}) \Rightarrow (\mathrm{ii}) \Rightarrow (\mathrm{i}).$$

(i) $\Rightarrow$ (iv). If, for any $a \in R$, $a^r = 0$ then $(a^2)^r$, $(a^3)^r$,... are all zero and the chain $aR \supseteq a^2R \supseteq a^3R \supseteq \cdots$ implies by assumption that for some positive integer m, $a^mR = a^{m+1}R$. Suppose $a^m = a^{m+1}x$ for a certain x in R. Then $a^m(1 - ax) = 0$ together with $(a^m)^r = 0$ implies that $ax = 1$. Now it is easy to see that $a^l = 0$.

(iv) $\Rightarrow$ (v). For a fixed nonzero element h in H, the mapping $a \rightarrow ha$ of R onto H induces the isomorphism $R/h^r \simeq H$. Thus, by minimality of H it follows that h^r is a maximal right ideal. Let $x \in h^r$ be an arbitrary element. Then $x^l \neq 0$ and by (iv), $x^r \neq 0$. Using 1.2, we have $x \in N$. Thus $h^r \subseteq N$ and since h^r is maximal, $h^r = N$.

(v) $\Rightarrow$ (vi). Let h be a fixed nonzero element of H. For any $a \in R$, consider $g_a : H \rightarrow H$ defined by $g_a(hx) = hax$ for arbitrary hx in $H = hR$. If $a \in N$, then $ha = 0$ by (v) and hence g_a is the zero map. If $a \notin N$, then ha is a nonzero element of H and by (v), $h^r = (ha)^r$ which implies that g_a is in this case, a well-defined, one to one map. It can be easily seen that it is a right R-homomorphism. Now the map $g : R/N \rightarrow D$ defined by $g(a + N) = g_a$ is the desired ring isomorphism between R/N and D.

(vi) $\Rightarrow$ (iii). Since D is a division ring, (vi) implies in particular that N is a maximal left ideal. If $h \neq 0$ and $h \in H$, then the mapping $a \rightarrow ah$ of R onto Rh is a left R-homomorphism which induces the isomorphism $R/h^l = R/N \simeq Rh$. This implies that Rh is a minimal left ideal.

(iii) $\Rightarrow$ (ii). Since we have already seen that $R/N \simeq Rh$, (iii) implies that N is a maximal left ideal. However, N is a twosided ideal and consequently R/N is a division ring. Now it is clear that N is also a maximal right ideal.

(ii) $\Rightarrow$ (i). Since N is a maximal right ideal and also a two-sided ideal, R/N is a division ring. Also, since R satisfies a.c.c. on right ideals, N is nilpotent. Say, $N^n = 0$ and $N^{n-1} \neq 0$ for some positive integer n. For $i = 0, 1, 2,..., n - 1$, each of N^i/N^{i+1} is a right R-module as well as a R/N right vector space with the property that every R-submodule is an R/N-subspace and vice versa. Since R has right a.c.c., each N^i/N^{i+1} is now a finitely generated vector space over R/N. The composition series for each N^i/N^{i+1} as R/N vector spaces can now be combined to give a composition series for R_R which implies that R has d.c.c. on right ideals.

Finally, to see that R is local, we note that N coincides with the set of zero divisors of R because $N = \{a \in R : a^r \neq 0\}$ and we have seen that for every zero divisor a, $a^r \neq 0$. Now, in a ring with d.c.c. on one-sided ideals, each nonunit is a zero divisor. Consequently, N in this case coincides with the set of nonunits of R. By one of the equivalent definitions of a local ring, this implies that R is local with N as the unique maximal right ideal.

Remark. It may be noted that all of the implications except (ii) $\Rightarrow$ (i) above have been proved without using a.c.c. on right ideals of R. The author feels obliged to the referee for suggesting this simplified proof.

The following example shows that there do exist RSI rings which satisfy a.c.c. on right ideals but none of the equivalent conditions of the above theorem.

2.2 EXAMPLE. Let F denote the field of rational functions in a countably infinite number of indeterminates $x_2, x_3, \ldots, x_n, \ldots$ and let $R_1 = F[x_1]$ be the ring of polynomials in x_1 with coefficients in the field F. Define a monomorphism σ and an epimorphism τ on R_1 to F by $\sigma[f(x_1, \ldots, x_n)] = f(x_2, \ldots, x_{n+1})$ and $\tau[f(x_1, x_2, \ldots, x_n)] = f(0, x_2, \ldots, x_n)$ respectively. The ring R defined by $(R, +) = R_1 \oplus F$ and by $(a, b)(c, d) = (ac, b\tau(c) + \sigma(a)d)$ for $(a, b), (c, d) \in R$ has the above stated properties.

In fact, if h denotes the element $(0, 1)$ of R and (a, b) an arbitrary nonzero element; either $a \neq 0$ in which case $(a, b)(0, \sigma(a)^{-1}) = h$, or $a = 0$ and $b \neq 0$ so that $(0, b)(c, 0) = h$ where $c \in R_1$ is chosen such that $\tau(c) = b^{-1}$. In any case, h belongs to the right ideal generated by (a, b). Thus R is RSI with heart hR. Next, the mapping $(a, b) \to a$ of R onto R_1 induces the isomorphism $R/hR \simeq R_1$. Since R_1 is a (commutative) ring with a.c.c. but not the d.c.c. on its ideals, it follows that R satisfies a.c.c. but not the d.c.c. on its right ideals.

Remark. The ring R consrtucted above is a right noetherian right uniform ring with identity, for which the elements not in $N = \{a \in R : a^r \neq 0\}$ are not all left regular. The question as to whether such rings exist was raised by Feller in [6, p. 137]. The element $(x_1, 0)$ of R does not belong to N and yet it is not left regular because $(0, 1)(x_1, 0) = (0, 0)$. In fact, the equivalence (i) $\Leftrightarrow$ (iv) of Theorem 2.1 above says that for a right noetherian RSI ring, elements not in N will all be regular iff the ring is right Artinian.

3. RINGS WITH D.C.C. ON RIGHT IDEALS

Though examples of RSI rings with a.c.c. but not the d.c.c. on right ideals exist, Theorem 2.1 suggests that RSI rings with d.c.c. on right ideals form an important class of rings. We will now prove for the class of rings with d.c.c. on right ideals, an analogue of McCoy's theorem [9, p. 382] which was proved for the commutative case.

3.1. THEOREM. *Let R be a ring with d.c.c. on right ideals. Then R is* RSI *iff the following conditions hold.*

(i) *The set Z of zero divisors of R forms an ideal*;

(ii) $Z^l = \{a \in R : aZ = 0\}$ *is a principal right ideal; say* $Z^l = xR$ *where* $x \neq 0$;

(iii) *For any* $a \in Z$, $a \notin Z^l$; *there exists* $b \in Z$, $b \notin Z^l$ *such that* $ab = x$.

Proof. ($\Rightarrow$). If R is RSI, then by Theorem 2.1 it is local with N as the set of nonunits. Since R satisfies d.c.c. on right ideals, N coincides with the set of zero divisors Z of R which implies (i). Now, by definition of N, $H^l = N = Z$. If $a \in R$, $a \neq 0$; then by the isomorphism $R/a^r \simeq aR$, a belongs to a minimal right ideal iff a^r is a maximal right ideal. Since H and Z are the unique minimal and the unique maximal right ideals, respectively, $a \in H$ iff $a^r = Z$ which implies $Z^l = H = xR$, where x is a nonzero element of H. Lastly, if $a \in Z$, $a \notin H$; then $x \in aR$ implies that $x = ab$ for some $b \in R$. Clearly $b \notin H$ because $aH = 0$. Also we must have $b \in Z$ because otherwise b is a unit which implies $a = xb^{-1} \in H$.

($\Leftarrow$). Since R has d.c.c. on right ideals, Z = set of zero divisors = set of nonunits and hence R is local with Z as the unique maximal right ideal. We will prove that R is RSI by showing that every nonzero principal right ideal of R contains x. Accordingly, let $a \neq 0$ be any element of R. If $a \in Z^l$, then by (ii), $a = xr$ for some $r \in R$. r cannot be in Z because $xZ = 0$ and $a \neq 0$. Thus r is a unit which implies $x = ar^{-1} \in aR$. If $a \notin Z$ then a is a unit and $x \in aR$ is trivial. Finally, if $a \in Z$ and $a \notin Z^l$, then by (iii) $x \in aR$. Thus we have proved R is RSI with heart $xR = Z^l$.

4. Completely Indecomposable Modules and Rings

A module M_R over a ring R (with identity) is said to be completely indecomposable if

(i) M satisfies d.c.c. on R-submodules and contains a unique minimal R-submodule;

(ii) For some ring A, ${}_AM_R$ is a bimodule where M satisfies d.c.c. on A-submodules and contains a unique minimal A-submodule. Completely indecomposable modules over noncommutative rings were first introduced by Feller [5, p. 349], and Morita [10, p. 121] later showed that the notion of complete indecomposability is closely related to injectivity.

We first consider a slightly more general situation.

4.1. Theorem. *Let* ${}_AM_R$ *be a bimodule such that*

(i) M *satisfies a.c.c. on* A*-submodules and* R*-submodules,*

(ii) M *is subdirectly irreducible as* A*-module and as* R*-module,*

(iii) *M is faithful as R-module and R satisfies a.c.c. on right ideals*;

then M satisfies d.c.c. on R-submodules and R is a local ring with d.c.c. on right ideals.

Proof. As in Lemma 1 of [5, p. 349], the A-heart M_0 of M coincides with the R-heart. Let $m \neq 0$ be any element of M_0 so that $M_0 = Am = mR$. If $P = \{x \in R : mx = 0\} = \{x \in R : M_0 x = 0\}$, then $f_m : R \to M_0$ defined by $f_m(x) = mx$ is a R-homomorphism with ker P and accordingly we have the R-module isomorphism $R/P \cong M_0$. This implies that P is a maximal right ideal. Also, being the right annihilator of M_0, it is a two-sided ideal in R and therefore R/P is a division ring.

We will now show that P is a nilpotent ideal in R. Thus, if $x \in P$, consider $x^l = \{p \in M : px = 0\}$. Then $(x)^l \subseteq (x^2)^l \subseteq \cdots$ is a chain of A-submodules of M and, consequently, for some integer k, $(x^k)^l = (x^{k+1})^l = \cdots$. Also, since $M_0 \subseteq x^l \subseteq (x^k)^l$, this implies in particular that $(x^k)^l \neq 0$. If $a \in (x^k)^l \cap Mx^k$, then $ax^k = 0$ and $a = px^k$ for some $p \in M$. This implies that $0 = ax^k = px^{2k}$, i.e., $p \in (x^{2k})^l = (x^k)^l$ and hence $a = px^k = 0$. Thus $(x^k)^l \cap Mx^k = 0$, and by (ii) we have $Mx^k = 0$. Since M_R is faithful, we then have $x^k = 0$ and thus P is a nil ideal. Also R has a.c.c. on right ideals which implies that P is nilpotent. Now, as in the proof of (ii) $\Rightarrow$ (i) in Theorem 2.1, the composition series for R/P, P/P^2, P^2/P^3 ,... as right vector spaces over R/P give rise to a composition series for R_R and hence R satisfies d.c.c. on right ideals. Also, by applying 54.8 of [2, p. 371] and observing that in a RSI ring there are no nontrivial indempotents, we conclude that R is local. Since M is a finitely generated R-module and R has d.c.c. on right ideals, by [2, Example 18, p. 59], M has d.c.c. on R-submodules. This completes the proof.

We note that if we drop the assumption M_R is faithful, then the conclusion can be modified to R/I being local with d.c.c. on right ideals where $I = \{x \in R : Mx = 0\}$.

4.2. COROLLARY. *Let ${}_AM_R$ be a bimodule such that*

(i) *M has a.c.c. on A-submodules and on R-submodules*;

(ii) *R is right noetherian and A is left noetherian*;

(iii) *M is A-subdirectly irreducible and R-subdirectly irreducible,*

then A/P and R/I are local rings with d.c.c. on left and right ideals, respectively, and M_R is completely indecomposable where $P = \{a \in A : aM = 0\}$ and $I = \{r \in R : Mr = 0\}$.

Proof. We can apply Theorem 4.1 to M as a left A/P module and as a right R/I module because M is a faithful module over each of these rings and in addition A/P and R/I have a.c.c. on left and right ideals, respectively.

Another important consequence of 4.1 can be obtained by considering the case where $M = A = R$.

4.3. Corollary. *If R is a* RSI *ring with a.c.c. on left and right ideals, then R is a local ring with d.c.c. on left and right ideals.*

It may be noted that whereas Example 2.2 above shows the existence of RSI rings with a.c.c. but not the d.c.c. on right ideals, by the above corollary RSI rings with a.c.c. on both sides have d.c.c. on both sides too.

A ring R is completely indecomposable if R_R is a completely indecomposable module; or equivalently, R is a left and right uniform ring with d.c.c. on left and right ideals. As an application of 4.3 above, we can characterise completely indecomposable rings by

4.4. Proposition. *A ring R is completely indecomposable iff R is a right and left subdirectly irreducible ring with a.c.c. on left and right ideals.*

Morita's Theorem [10, p. 122] applied to the case of the bimodule ${}_RR_R$ says that a ring R is completely indecomposable iff it is a quasi-Frobenius local ring. Thus by Proposition 4.4 above, a ring R is a quasi-Frobenius local ring iff it is a right and left subdirectly irreducible ring with a.c.c. on right and left ideals. To complete the picture, we give below an example of a RSI ring with d.c.c. on left and right ideals which is not completely indecomposable. This will show that the hypothesis of left subdirect irreducibility in Proposition 4.4 is necessary. Also the same example[1] answers a question raised by P. M. Cohn in [3, p. 75]. Cohn poses the question whether in a noetherian ring R, the conditions

1. R_R is self injective and
2. every right ideal is an annihilator are equivalent.

We will show that the ring R of Example 4.5 satisfies condition 2. It cannot satisfy Condition 1 because then R, being quasi-Frobenius and local, will be completely indecomposable.

4.5. Example. Let F denote the field of all rational functions in an indeterminate x with coefficients in the field of rational numbers. Let σ denote the isomorphism of F into F given by $\sigma[f(x)] = f(x^2)$ for any element $f(x) \in F$. Define $R = \{(a, b) : a, b \in F\}$ in which equality and addition of elements is componentwise and multiplication obeys the law $(a,b)(c,d) = (ac, bc + \sigma(a)d)$. R is then a RSI ring because $H = \{(0, b) : b \in F\}$ is the only right ideal of R other than 0 and R. Obviously R satisfies d.c.c. on right ideals. By a routine computation it can be verified that $R \supset H \supset L_1 \supset 0$ where $L_1 = \{(0, \sigma(a)) : a \in F\}$ is a composition series for ${}_RR$ and hence R also satisfies d.c.c. on left ideals.

[1] The referee has kindly informed the author that a similar construction has been used by J. E. Björk to give a counter-example to Cohn's question (unpublished).

If $L_2 = \{(0, \sigma(a)x) : a \in F\}$ then L_2 is a left ideal of R such that $L_1 \cap L_2 = 0$. This implies that R cannot be left subdirectly irreducible and, consequently, it is not completely indecomposable.

Obviously, 0 and R are annihilator right ideals. Also $H^l = H^r = H$ as can be verified directly or deduced from Theorem 2.1 and the definition of N in an RSI ring. Thus $H^{lr} = H$ and R is a ring satisfying Condition 2 above but not Condition 1.

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References

1. G. Birkhoff, Subdirect unions in universal algebra, *Bull. Amer. Math. Soc.* **50** (1944), 764–768.
2. C. W. Curtis and I. Reiner, "Representation Theory of Finite Groups and associative Algebras," Interscience Publishers, Inc., New York, 1962.
3. P. M. Cohn, "Morita Equivalence and Duality," Queen Mary-College, Mathematics notes, London.
4. N. Divinsky, Commutative subdirectly irreducible rings. *Proc. Amer. Math. Soc.* **8** (1957), 642–648.
5. E. H. Feller, The lattice of submodules of a module over a noncommutative ring, *Trans. Amer. Math. Soc.* **81** (1956), 342–357.
6. E. H. Feller, Noetherian modules and noetherian injective rings, *Tohoku Math. J.* **17** (1965), 130–138.
7. E. H. Feller, A type of Quasi-Frobenius ring, *Canad. Math. Bull.* **10** (1967), 19–27.
8. J. Lambek, "Lectures on Rings and Modules," Blaisdell Publishing Co., Waltham, Mass., 1966.
9. N. H. McCoy, Subdirectly irreducible commutative rings. *Duke Math. J.* **12** (1945), 381–387.
10. K. Morita, Duality for modules and its application to the theory of rings with minimum condition, science report of *Tokyo Kyoiku Daigaku* **6** (1958), 83–142.